Lecture 13

7.5 - Strategies for Integration

We have some basic forms for integrals:

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	② $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
③ $\int e^x\,dx = e^x + C$	④ $\int a^x\,dx = \frac{a^x}{\ln a} + C$
⑤ $\int \sin x\,dx = -\cos x + C$	⑥ $\int \cos x\,dx = \sin x + C$
⑦ $\int \sec^2 x\,dx = \tan x + C$	⑧ $\int \csc^2 x\,dx = -\cot x + C$
⑨ $\int \sec x \tan x\,dx = \sec x + C$	⑩ $\int \csc x \cot x\,dx = -\csc x + C$
⑪ $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$	⑫ $\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + C$
⑬ $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	⑭ $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
⑮ $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$	⑯ $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$ $(a>0)$

We have a variety of techniques we can apply. If it doesn't immediately fit one of the basic forms, we use an appropriate technique:

① Use a u-sub to turn it into a basic form.

e.g. $\int e^x \sin(e^x)\,dx$ $\quad (u=e^x)$

② Simplify the integrand/rewrite it in another way

e.g $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$, $\int \frac{\sin\theta \cot\theta}{\sec\theta}\,d\theta$

③ Apply trig integral tricks (c.f. lecture 9) if relevant

e.g: $\int \cos^n x \sin^m x\,dx$, $\int \sec^m x \tan^n x\,dx$, $\int \sin(mx)\sin(nx)\,dx$,

$\int \sin(mx)\cos(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$

④ If the integrand is a rational function, try partial fraction decomposition

⑤ Look for a trig sub if there is a term of the form $\sqrt[n]{\pm x^2 \pm a^2}$ (c.f. lecture 10). If there is an expression of the form $\sqrt[n]{ax+b}$, use a rationalizing substitution, $u=\sqrt[n]{ax+b}$. This could also work for more general things like $\sqrt[n]{g(x)}$ using $u=\sqrt[n]{g(x)}$.

⑥ Try integration by parts.

⑦ Manipulate the integrand into a more managable form

e.g.: $\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$

$\int \frac{1}{1+e^x}\,dx = \int \frac{e^x}{e^x+e^{2x}}\,dx$

(could also write $\sec x = \sec x \cdot \frac{\sec x+\tan x}{\sec x+\tan x}$)

⑧ It may take multiple steps

e.g. ⓐ u-sub + partial fractions + another u-sub

ⓑ u-sub + integration by parts

ⓒ integration by parts + trig sub, etc...

Here is a list of several integrals. Outline how you would approach the integral:

(1) $\int \ln x \, dx$

(2) $\int \tan x \, dx$

(3) $\int \sin^3 x \cos x \, dx$

(4) $\int \frac{1}{\sqrt{25 - x^2}} \, dx$

(5) $\int \sec x \, dx$

(6) $\int e^{\sqrt{x}} \, dx$

(7) $\int \sin(7x) \cos(4x) \, dx$

(8) $\int \cos^2 x \, dx$

(9) $\int \frac{1}{x^2 - 9} \, dx$

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Additional integrals:

(1) $\displaystyle\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$

(2) $\displaystyle\int \sqrt{\frac{2-x}{2+x}}\,dx$

(3) $\displaystyle\int \ln(x^2-1)\,dx$

(4) $\displaystyle\int 35\arctan\sqrt{x}\,dx$

(5) $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx$

(6) $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx$

It is possible that we just cannot compute the integral. For example,

$$\int e^{x^2}\,dx, \int \frac{e^x}{x}\,dx, \int \frac{1}{\ln x}\,dx, \int \frac{\sin x}{x}\,dx$$

cannot be computed in terms of elementary function. We can approximate them using varias methods. Another option is to use Taylor series expansions (c.f. chapter 11).